Solutions vs. Roots vs. Zeros

**Solutions** are any values that can be entered into a function which will calculate to a positive result. These are called solutions because they don’t always have to result in a zero.

**Roots** are the \( x \) - intercepts of a function which will always result in \( f(x) = 0 \), meaning, \( y = 0 \).

**Zeros** another name for roots.

The Quadratic Formula

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

The Quadratic Formula is used to find the roots/zeros of a quadratic function. When substituted into the function, the roots are the solution if the result is true.

Example and Counterexample to the Quadratic Formula and its roots:

Solve \( f(x) = 2x^2 - 8x - 10 \)

\[
x = \frac{(-8) \pm \sqrt{(-8)^2 - 4(2)(-10)}}{2(2)}
\]

\[
x = \frac{8 \pm \sqrt{64 + 80}}{4}
\]

\[
x = \frac{8 \pm \sqrt{144}}{4}
\]

\[
x = \frac{8 \pm 12}{4}
\]

\[
x = \left\{ \frac{8 + 12}{4}, \frac{8 - 12}{4} \right\}
\]

\[
x = \left\{ \frac{20}{4}, \frac{-4}{4} \right\}
\]

\[
x = \{5, -1\}
\]

<table>
<thead>
<tr>
<th>if ( x = -1 )</th>
<th>if ( x = 5 )</th>
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</thead>
<tbody>
<tr>
<td>( f(-1) = 2(-1)^2 - 8(-1) - 10 )</td>
<td>( f(5) = 2(5)^2 - 8(5) - 10 )</td>
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<tr>
<td>( = 2(1) + 8 - 10 )</td>
<td>( = 2(25) - 40 - 10 )</td>
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<tr>
<td>( = 2 + 8 - 10 )</td>
<td>( = 50 - 40 - 10 )</td>
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<td>( = 10 - 10 )</td>
<td>( = 10 - 10 )</td>
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<tr>
<td>( = 0 )</td>
<td>( = 0 )</td>
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FINDING THE Y-INTERCEPT
The y-intercept is found by substituting \( x = 0 \). In this case \( 2(0)^2 - 8(0) - 10 = -10 \). So, \((0, -10)\)

Finding the rest of the points

Up 'til now, we have four points; two from the quadratic formula, one from the vertex and one from the y-intercept. Your graph should start to look like →

If we use the vertex for the axis of symmetry, we can build a “mirror”. Drawing a vertical line through the vertex should then make your graph look like this →

A completed graph would look like this →

FINDING THE VERTEX

\[
\left( -\frac{b}{2a}, f\left( -\frac{b}{2a} \right) \right)
\]

The vertex allows us to find the axis of symmetry which, in turn, allows us to create a “mirror” and cut down half the work of finding points to graph a quadratic function. In the case of \( f(x) = 2x^2 - 8x - 10 \), the vertex would be:

\[
-\frac{b}{2a} = -\frac{-8}{2(2)} = \frac{8}{4} = 2
\]

\[
f\left( -\frac{b}{2a} \right) = 2(2)^2 - 8(2) - 10
\]

\[
= 2(4) - 16 - 10
\]

\[
= 8 - 16 - 10
\]

\[
= -8 - 10
\]

\[
= -18
\]

The vertex is located at \((2, -18)\)
Demystifying Quadratics

Solving by Factoring
A quadratic equation might be simple enough to factor. Take, for example, \(x^2 - 5x + 6 = 0\). In factored form, this can be rewritten as \((x - 2)(x - 3) = 0\). Therefore, either \(x - 2 = 0\) or \(x - 3 = 0\). Our solutions are \(x = \{2, 3\}\) as those are the values that result in a value of 0.

Illegal Factoring vs Quadratic Formula
Sometimes a quadratic function isn’t that easy to solve. For example, \(6x^2 - 13x + 5 = 0\) can be a real challenge.

Using the Quadratic Formula
\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]
\[
= \frac{-(-13) \pm \sqrt{(-13)^2 - 4(6)(5)}}{2(6)}
\]
\[
= \frac{13 \pm \sqrt{169 - 120}}{12}
\]
\[
= \frac{13 \pm \sqrt{49}}{12}
\]
\[
= \frac{13 \pm 7}{12}
\]
\[
= \frac{20}{12} \text{ OR } \frac{6}{12}
\]
\[
= \frac{5}{3} \text{ OR } \frac{1}{2}
\]

Using Illegal Factoring w/Steps
\[
6x^2 - 13x + 5 = 0 \quad \text{Given Function}
\]
\[
x^2 - 13x + 30 = 0 \quad \text{Multiply first term to third}
\]
\[
(x - 10)(x - 3) = 0 \quad \text{Factor Normally}
\]
\[
\left(\frac{x - \frac{10}{6}}{\frac{1}{6}}\right)\left(\frac{x + \frac{3}{6}}{\frac{1}{6}}\right) = 0 \quad \text{Divide out the original 6}
\]
\[
\left(\frac{x - \frac{5}{3}}{\frac{1}{3}}\right)\left(\frac{x + \frac{1}{2}}{\frac{1}{2}}\right) = 0 \quad \text{Reduce any fractions. The solution can be found here}
\]
\[
(3x - 5)(2x + 1) = 0 \quad \text{Factored Form}
\]

Either \((3x - 5) = 0\) OR \((2x + 1) = 0\) Because if \(ab = 0\) then either \(a = 0\) or \(b = 0\)
\[
\quad \quad 3x - 5 = 0 \quad \quad 2x + 1 = 0
\]
\[
\quad \quad 3x = 5 \quad \quad 2x = -1
\]
\[
\quad \quad x = \frac{5}{3} \quad \quad x = -\frac{1}{2}
\]

Solving by Square Roots
Sometimes, we don’t need the Quadratic Formula to solve a quadratic equation. Something as simple as:

Solve: \(\frac{1}{6}x^2 = 24\)

\[
6 \cdot \frac{1}{6}x^2 = 24 \cdot 6
\]
\[
x^2 = 144
\]
\[
x = \pm 12
\]
**Vertex Form**

Vertex Form is a different way of writing quadratics. Rewriting a quadratic from standard form to vertex form requires **completing the square**.

Rewrite $x^2 + 6x + 11 = 0$ in vertex form

\[
x^2 + 6x + 11 = 0 \quad \text{Given function}
\]

\[
x^2 + 4x = -11 \quad \text{Add/Subtract the constant to the other side of the equal sign (in this case, 11). We do this because that number is usually not a square and we need it to be.}
\]

\[
x^2 + 6x = -11 \quad \text{Complete the square. This means we need to take the } b \text{ value, divide it in half and then square it.}
\]

\[
x^2 + 6x + (\frac{6}{2})^2 = -11 + (\frac{6}{2})^2 \quad \text{Simplify and be sure it’s done on both sides}
\]

\[
x^2 + 6x + 9 = -11 + 9 \quad \text{(x + 2)}^2 = x^2 + 4x + 4 \quad (x - 3)^2 = x^2 - 6x + 9 \quad (x + 4)^2 = x^2 + 8x + 16 \quad (x - 5)^2 = x^2 - 10x + 25
\]

Wait! What? Why does this work? Check out some other examples of much easier problems →

In each case, the end number $c$ is the square of half the middle number $b$ because these are squares.

Moving on...

\[
x^2 + 6x + 9 = -2 \quad \text{Simplify right side}
\]

\[
(x + 3)^2 = -2 \quad \text{Rewrite left as a square}
\]

\[
(x + 3)^2 + 2 = 0 \quad \text{Solve for zero and voilà! We have vertex form.}
\]

To solve **using** completing the square or from vertex form, we need to go back a step

\[
(x + 3)^2 = -2 \quad \text{Add the constant (if in vertex form)}
\]

\[
\sqrt{(x + 3)^2} = \sqrt{-2} \quad \text{Take the square root of both sides}
\]

\[
x + 3 = \pm i\sqrt{2} \quad \text{Simplify}
\]

\[
x = -3 \pm i\sqrt{2} \quad \text{Ick…it’s ugly, but there is the solution.}
\]

Why tho? In vertex form $f(x) = a(x - h)^2 + k$, $(h,k)$ is the vertex. In our case above, $(-3, 2)$. Other features are easily identifiable from vertex form such as $a$ for amplitude, etc. But we’re not there yet.